Logic Course

Overall Goals for this Course

The intent of this 'course' is to introduce students to the very nature of mathematics. Mathematical truth is not empirical truth: it does not depend on experimentation and observation. Mathematical truth is not religious truth: it does not depend on faith or explore the ethics of human interactions.

Rather, mathematical truth is logical truth. What this means is that it explores the relationships among statements, and in particular which statements imply other statements.

The nature of implication as a logical relationship is subtle, and it will take students some years to fully appreciate it. However, central elements of the notion of implication are embedded in our intuition, and we can work to refine students' intuitions about these logical relationships.

In particular, younger students very naturally accept the fact that new information can be derived from old: that if we are told certain things, then certain other things follow. While a deeper description of this phenomenon may elude them, they can practice and sharpen their intuitions with interesting puzzles, stories and games.

Many of these puzzles do not, on the surface, resemble traditional mathematics. It may come as a surprise to the students (and even to their parents, often!) that in working these puzzles, they are practicing some of the same skills that they will be harnessing in learning more traditional mathematical material.

The notion that if statement A is true, then statement B is also true (“if A then B”) is one that comes naturally to students. The subtleties come later, when we ask under what circumstances A does not imply B, or when they must distinguish this statement from “if B then A”, or when they explore the notion of indirect proof (proof by contradiction). Several examples will be encountered in the problems and puzzles offered below.

Most of these problems are almost content-free. That is, students need know very little mathematical background to work them. However, we have taken the opportunity of talking about logic to introduce some very specific logical principles that are often used in mathematics. These include the pigeonhole principle, invariants, parity, and isomorphisms. Again, these concepts are very subtle, and students will learn much more about them in advanced work. But an early introduction can only benefit them as they grow intellectually.

We have often included advanced notes for the teacher, which are not always appropriate for the student. It is important for the teacher to know more than she or he is trying to teach. (If a teacher is teaching addition, he or she should certainly know something about what it leads to, including multiplication—and probably much more!)

Our point of view about elementary mathematics, including some of these amusing and simple problems, is that it must be seen as 'embedded' in much more serious mathematics. Mathematics as a discipline has a certain unity: every small part of it betrays the structure of the whole. And that structure is its logical structure.

We hope your students enjoy solving these puzzles as much as we have enjoyed selecting them.

<u>Day One:</u>
Animal Puzzles (Introduction to Logical Reasoning with Cryptarithms)
The Game of Giotto (Practice with Pure Logical Reasoning)
Lewis Carroll Puzzles (Proof by Contradiction)
Cryptogram Puzzle (More Proof by Contradiction)

<u>Day Two:</u>
Giotto Puzzles (Analyzing the Logical Structure of Giotto)
Watermelon Language (Logic Applied to Number Systems)
Jittery Soldiers (An Introduction to Invariants)
The Black and Red Problem (Introduction to Parity)

<u>Day Three:</u>
Parity Problem Set
Discuss solution to Black and Red Problem (A Surprising Connection to Parity)
Ginger's Pigeons (Proofs with the Pigeonhole Principle)
The Mouse-and-Cheese Problem (Another example of Parity)

<u>Day Four:</u>
Nim (An Introduction to Mathematical Induction)
Two-Row Nim and One-Piece Chess (Induction and the Idea of Isomorphism)
Leap Frog (Another Example of the Use of Invariants)

<u>Day Five:</u>
The Boys and Girls Problem (Another Example of the Use of Invariants)
Hidden Cards Puzzle (Practice with Logical Deductions)
Magic Squares and 15 Game (More Practice with Invariants and Isomorphism)

DAY 1

Goals for the Day-
Students will be introduced to logic and the nature of 'if-then' reasoning. Students will also be introduced to a specific kind of "proof" – proof by contradiction.

Animal Puzzles

Background:
We use these puzzles (called *cryptarhythms*) to introduce logical reasoning. The point is that while the puzzles look mathematical, the hard part here is the logic – not the arithmetic.

In a running math circle, it can be nice to begin every day with a few cryptarithms. This will allow students to begin working on something familiar. The activity is 'modular', and so the timing is not well-defined. For example, early students can work the puzzles for fifteen minutes while waiting for everyone to arrive, or a math circle could use them as the day's activity. Finally, the puzzles don't always require sustained thought. Students can disengage and come back to them later on.

To begin, put puzzles on the board one or two puzzles at a time and have the class work on them together. If students want to do more, see the websites listed (and there are many others), which give more difficult examples.

As another extension, encourage the students to make their own puzzles. They are not easy to make. A classic cryptarhythm is considered elegant if it has a unique solution. But for beginning students, the educational purpose is served if we relax this condition a bit.

For example, the cryptarithm AB + BC = CD has far too many solutions to be of interest, except perhaps as an exercise. On the other hand, the cryptarhithm AB + BC = CA has no solutions at all. Again, this is not very satisfying as an introductory problem. But again, a proof that it has no solutions might make a good exercise for students later on.

Note, too, that these really are problems. The statement is easily comprehended. The goal is well-defined. But there is no obvious algorithm, or sequence of steps, that will surely solve the problem (short of trying every value of the digits, which humans generally do not want to do). There is not even one pre-determined method of attack. These problems make students think.

Each cryptarithm puzzle below, represents an arithmetical statement. Within each puzzle, each letter represents a unique digit, and no two letters represent the same digit. In other words, in a single puzzle, a P could represent the digit 4, and if so it represents a 4 everywhere in the puzzle, and no other letter will have that value. But the value of P can be different in different puzzles. If we write AB, we mean the integer with tens digit A and units digit B. That is, juxtaposition of letters denotes place value, and not multiplication. Multiplication, either of single digits or multi-digit numbers, is indicated with a 'x'. We assume, for this sort of puzzle, that there are no leading zeroes (zeroes to the left of the numeral).

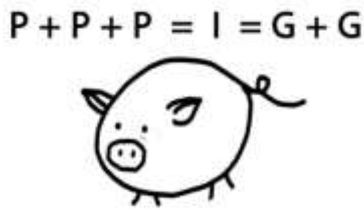
A. Introductory puzzles.

1.

W+O = O F

Solution: We know that a one-digit number plus a one-digit number gives us a two-digit number, whose first digit is 1, so O must be 1. If O = 1, then W = 9, because otherwise the sum would be one digit and not two. 9 + 1 = 10.

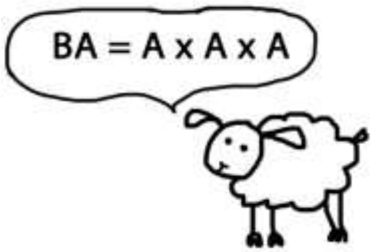
2.



Solution: I must be a single-digit number divisible by 2 and 3. So I must be 6, which means P = 2 and G = 3. Students may reason about it differently. Some will reach the same solution by plugging in numbers and ruling out possibilities. But after they have solved the puzzle, it is useful to point out the more elegant solution we have given.

Notice that this puzzle uses completely different reasoning that the previous puzzle.

3.



Solution: $A \times A \times A$ has only two digits. A quick check shows that the only candidates are $3 \times 3 \times 3$ and $4 \times 4 \times 4$, of which only $4 \times 4 \times 4 = 64$ works (because BA must end with the digit A).

4.

C + CA = A T T

Solution: The only way that the sum of a single-digit number and a double-digit number can be a 3-digit number is for C to be 9 and A to be 1. 9 + 91 = 100.

5.

$$\begin{array}{r} ME \\ +ME \\ \hline BEE \end{array}$$

Solution: B = 1 because it cannot be larger than 1 (which is similar to the reasoning in the CAT puzzle.) E = 0 because E + E ends with E. (The other possibility would be 5 + 5 = 15, but we run into a contradiction later.) The solution is 50 + 50 = 100.

6.

HH + HH = OOT

Solution: O cannot be any larger than 1, so it must be 1. Since H + H in the 10s place gives us a 1, there must be a 1 carrying over. With a little trial and error, it is clear that H must be 5. 55 + 55 = 110.

7.

Solution: An ono is another name for a wahoo, which is a fish. O = 1. N must equal 0 because of the first column on the right. 91 + 10 = 101.

8.

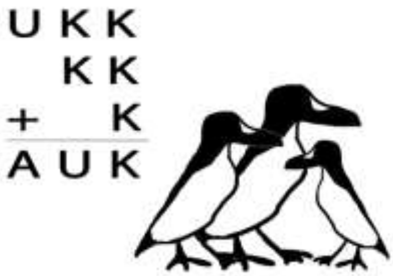
$$A + A + A + A + A + A + A + A + A + P = E$$

Solution: It is not hard to see that A cannot be too large, since nine A's plus a P is still a one-digit number. In fact, if A were as much as 2, the sum on the left would be a two-digit number. So A = 0 or A = 1. If A were 0, then P and E would represent the same digit. So A = 1, and it is not hard to see that then E must be 9 and P must be 0.
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 0 = 9

9.

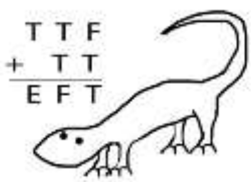
Solution: In this problem, A cannot be too small. In fact, if A were as small as 8, AA + A could not be a three-digit number. So A = 9, and ELK is 108. 99 + 9 = 108.

10.



Solution: An 'auk' is bird, similar in appearance to a penguin, but not closely related. K must be 5, because 3 x K gives us an K in the units place. (The other possibility is that K = 0, but that leads to problems.) From there we can work out the rest easily. The U must be a 1, so AUK = 215. Solution: 155 + 55 + 5 = 215.

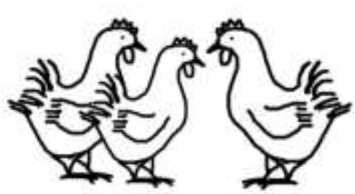
11.



Solution: An ‘eft’ is a life stage of a newt. F must be zero because F + T = T. Then T must be 5, because T + T ends with a zero (in the 10s place) Thus we have 550 + 55 = 605.

12. What is the product of the following single-digit numbers:

C x H x I x C x K x E x N x P x O x T x P x I x E x S



Solution: There are 10 different digits, so one must be zero, and the product is zero.

13. Challenge cryptarithm for those who may be working ahead of the class:

MA x CA = WWW

Solution: 37 x 27 = 999. WWW must be a multiple of 111 = 3 x 37. So either MA or CA must be 37. Suppose MA = 37. Then the other product (CA) cannot be too large, because the product is a three-digit number. In fact, 37 x 30 = 1110, which is too big. So CA = 1 or 2. A quick trial-and-error shows that C = 2.

Attentive students may note that the solution to this cryptarithm is technically not unique. We could have MA = 37, CA = 27, or vice versa. The cryptarhithm is *symmetric* in these two numbers.

More Cryptarithms

LEVEL 1:

1. **AB - BA = A**

2. **A + BB = ADD**

3. **BC + B = DAD**

4. **BC + DC = CAB**

LEVEL 2:

5. **HEN + E = EGG**

6. **DAD + DAD + DAD + DAD + DAD = GLAD**

7. Find all possible solutions:

   AB + BA = (A+B)(A+B)

SOLUTIONS

1. **AB - BA = A**

Solution: A must be a multiple of 9. (Play around with the difference of numbers of this form, and you'll see why.) In fact, since A is a single digit, it must be 9. The solution is 98 - 89=9.

2. **A + BB = ADD**

Solution: A is a carry from tens to units column, so A = 1, and it is not hard to see that the unique solution is 1 + 99 = 100.

3. **BC + B = DAD**

Solution: D is a carry from the tens place, so D = 1. Then it is clear that B must be a large digit, and in fact must be 9 (or the sum would not have three digits). We have 9C + 9 = 1A1, and we must have C + 9 = 11, so C = 2. Thus the unique solution is 92 + 9 = 101.

4. **BC + DC = CAB**

Solution: We know that C =1, since the sum of two two-digit numbers cannot be bigger than 198, and so B = 2. Now we have 21 + D1 = 1A2. D could be 8 or 9 to give us a 3-digit number. Since there are no repeating digits in CAB, D must be 8. 21 + 81 = 102.

5. **HEN + E = EGG**

Solution: Since the hundreds digit in the sum is different from H, there must be a carry from the tens to the hundreds column. Likewise, since the tens digit in the sum is different from E, there must be a carry from the units to the tens column.

Examining the tens column, and considering the carries, we have E + 1 = G + 10, so E = G + 9. Since E is a single-digit number, we must have E = 9, G = 0.

Now we have H9N + 9 = 900, and it is clear that H = 8, N = 1. The solution is: 891 + 9 = 900.

We have written the solution using the language of algebra, which makes it easy to understand. Students may get the answer, but struggle to express themselves. At this stage, if students have the right answer, we can assume that they have an idea of the reasoning behind the answer, and pull out of them whatever explanation they can offer. The process of expressing oneself mathematically develops over time.

6. **DAD + DAD + DAD + DAD + DAD = GLAD**

Five times D ends with a D, so D must be 5 or 0. In fact it must be a 5 because we don't allow leading zeroes. (Also, the sum is a 4-digit number.) We have a carry of 2 from the units to the tens column.

From the tens column we see that 5 × A + 2 ends in A. That means A could be 2 or 7.

If O is 2, we have 525 + 525 + 525 + 525 + 525 = 2625, and G = A, which is not allowed.
If O is 7, we have 575 + 575 + 575 + 575 + 575 = 2875, which is the solution.

7. **AB + BA = (A+B)(A+B)**

Solution: When you add AB and BA, you will get a multiple of 11 – no matter what value A and B take on. Since here we get a product that is a square number, the product must be 11 x 11. In fact, this equation holds true for any A and B whose sum is 11:

29 + 92 = (2+9)(9+2)

38 + 83 = (3 + 8) (8+3)

etc.

Therefore, we don’t know what A and B are, but we know A + B.

The Game of Giotto

Overview:

This is one of a number of puzzles and games drawing on mathematical linguistics. Students enjoy it, and it works well when played at the board with the whole class. Once students learn the game, it can be played in about 20 minutes, and so it can be used to fill little gaps of time at the end of a session. The game has many interesting aspects, but here we are using it to give students practice with logical reasoning and to begin to make that reasoning explicit.

Classroom Procedure:

(a) The game is best played on the board with the whole class participating.

(b) The instructor should think of a secret five-letter word with no repeated letters. (Below we have chosen the word BREAD.) Write the secret word on the chalkboard in a place that can be covered up—or on a piece of paper that is folded and placed securely away from the students.

(c) The Rules

The students must work together to guess your word. They do so by offering their own 'test' words, five letters each. You report to them the number of letters their word has in common with yours.

For example, suppose your secret word is BREAD and that they offer the following test words:

Test Word	Score
NIGHT	**0**

BRINK	**2**	
CHEEP	**1**	**(Only one E counts in the score.)**
DREAD	**4**	
BEARD	**5**	
BREAD	**Right!**	

(Note that you do not tell the students which letters the test word has in common with the secret word, nor anything about the order of the letters in the secret word.)

The game ends when the students offer your secret word as a test word. When they do offer your secret word as a test, you can very dramatically reveal it, showing them to be correct.

(d) Issues that arise in the game:

Be careful. An error in the count spoils the game very quickly. One technique is to (silently) say to one's self: is the B in the test word? Is the R in the test word? Is the E in the test word? And so on, counting the letters one by one.

A well-chosen secret word is important. Students quickly realize that a score of 0 is very valuable. (Although they are at first surprised that a low score is actually better than a high score!) A score of 0 eliminates letters and allows for cleaner tests of other letters. So you want them to get a score of 0 quickly. The most common letters in English words, in order, are ETAOI NSHRDLU, so it is best to use the vowel U and some combination of the letters NSHRDL together with letters that occur more rarely. Some good secret words are: CRUMB, BLUNT, CUBED and so on. It might be best to avoid the letter S, since many test words are plurals of four-letter words.

At some point, it will become clear that an 'elimination alphabet' is a good tool: students write down an alphabet, cross out letters that are eliminated, and circle letters that are confirmed. If students don't think of this, you should at some point suggest it to them. It will serve to structure their thinking.

(e) Classroom Discussion (during and after the game)

For the first few games, you can act as 'scribe', writing the results of the test words on the board. Get students to discuss the meaning of the test results: which letters are eliminated, which are definitely in the secret word, which letters need further testing, and so on.

What emerges is a notion of proof. Some students will 'play a hunch', guessing that a certain letter is in the word. Others will give arguments that it must be, or cannot be, in the secret word. These are all good ways of getting students to think logically: to make deductions from known statements. This concept need not be made conscious. It is enough that students have the experience of engaging in logical deduction. A bonus for you is that they will be discussing their logic, and you can overhear and evaluate their stage of development.

(f) Extensions

(1) Students join the teacher's "team" in thinking of a word and scoring.

Have a student come up and whisper a secret word to you. Then you play on the side of the student against the rest of the class. Check carefully that the test words are given the right score.

(2) Students play in teams against each other.

Divide the class into two groups. Each must devise a test word, which they show you. You write it down and conceal it. They now play against each other, with you as moderator (checking their test scores!). They take turns offering test words. The team that guesses the other team's word first is the winner.

When students are playing against each other, each team should select a captain to mediate disputes as to what the next test word should be. They should also select a scribe, who takes over your duties in recording the test words and making notes on the elimination alphabet. But be sure to double-check the scores of the scribes! An error will ruin the game.

(3) Giotto for experts

This game can be made more mathematical by considering each letter as a Boolean variable. Its value is 1 (present in the secret word) or 0 (absent in the secret word). One can then do 'algebra' on the test words. In the example above (CRUMB), we have BRINK = CRACK (both have a score of 2) . But in fact the second c in 'CRACK' doesn't contribute, so we have

BRINK = CRAK.

By cancellation of R's and K’s

BINK = CAK.

That is, there are the same number of letters in each group and so the score of either group is identical. We don't know if the score is 1 or 2 without seeing the secret word because the canceled letter may or may not be in the secret word. (In this case, we know it is in, so both BINK and CAK have a score of 1.)

Then we can cancel the K's, to get: BIN = CA.

Thus we know that both groups of letters have the same score. If we were able, say, to eliminate CA (but in this case we can't) then we would know that BIN are also eliminated. We can use this information in a number of ways.

But this insight is only for a group that loves the game and gets very far into its analysis. It is not necessary for novices to enjoy and learn from the game.

The game can also be played with double letters allowed in the secret word. This version is considerably more difficult. For example, if the secret word is CREED, here are some sample test words and scores:

Test Word	
SHAME	1 (One E)
SHEER	3 (both E's and the R)
EERIE	3 (two of the three E's and the single R)

For most groups, the single-letter variant is more than enough for fun and learning. Note that this game can be used to fill up a 15-minute gap, without any attached 'lesson'.

<u>Proof by Contradiction</u>

Proof by contradiction is a quintessentially mathematical phenomenon. We assume that a statement is true. If we can deduce either a contradiction or a statement that is clearly false from such an assumption, then we know we must have been wrong, and it is the original statement is in fact false. Students often use this sort of reasoning naturally and unconsciously, but it is useful to make it conscious as well.

A contradiction is a relationship among statements, which must be false, no matter whether the statements themselves are true or false.

Students are sometimes uncomfortable with a statement that is a contradiction. They begin to think that they have themselves done something wrong. It is useful to show them that in fact they have been using the idea of a contradiction in solving other puzzles.

For example, you can recall to them the APE puzzle:

$A + A + A + A + A + A + A + A + A + P = E$

Here we immediately saw that A cannot equal 0, because them P would equal E. That is, we assumed that $A = 0$, and found that $P = E$. This contradicts the rules of our puzzle, which stipulate that P cannot equal E. We cannot have $P = E$ (implied by our assumption) and also P not equal E (implied by the rules), so we have arrived at a contradiction. We can conclude that our original assumption is false, and A is not equal to 0. We then figured out that $A = 1$, etc.

We will see this kind of reasoning also occurring naturally in playing GIOTTO.

Indirect reasoning is often used to show that a certain situation cannot happen.

Example 1: Show that the cryptarithm BAA + BAA = EWE has no solution.

Solution: There must be a carry from the units to the tens column. To see this, we use indirect reasoning. If there were no carry, then in the units column, $A + A = E$, and in the tens column $A + A = W$, so W would equal E, which is against the rules.

So there is a carry from the units to the tens column. Likewise there is a carry from the tens to the hundreds column. (We are adding the same numbers, with a carry of 1.)

From all this we derive the contradiction that E is both even and odd. It is even because it is the last digit (or perhaps the only digit) of A + A. It is odd because there is a carry from the tens to the hundreds column, so E = B + B + 1.

Example 2: The four-digit number 4365 has an interesting property If we write it backwards, and add the original number to the reversed number, we get 9999: 4365 + 5634 = 9999.

a) Show that 2097 and 4185 have the same property.
b) Find two more four-digit numbers that have this property.
c) Find a six-digit number that has this property.
d) Find a three-digit number that has this property.

Solution to (d): There are no such three-digit numbers. Indeed, this problem is equivalent to solving the cryptarithms.

$$\begin{array}{cccc} & A & B & C \\ + & C & B & A \\ \hline & 9 & 9 & 9 \end{array}$$

In the units column, there cannot be a carry, since we cannot get a sum of 19 by adding two digits (the largest sum we can get is 18). But then in the tens column we have B + B = 9, which is impossible since B + B must be an even number.

Another way to see that there is no carry from the units column is to note that we are adding the same two digits in the hundreds column (with a possible carry), and we don't need a thousands column, so there cannot be a carry.

Follow-up problem: For even n, show that there are many n-digit numbers with the property we are discussing. But for odd n, there are never any such n-digit numbers.

Lewis Carroll Puzzles

Lewis Carroll is known as the author of Alice in Wonderland, but he also taught mathematics and logic. The first puzzle was taken from Lewis Carroll's diary (1894). Carroll created logic puzzles like these to train his students in logical reasoning. Like many wonderful math problems, they often seem to give too little information for a solution, but a unique answer emerges. Puzzles like these give students an appreciation of the power of logic.

A says B lies;
B says C lies.
C says A and B lie.

Who lies and who tells the truth?

Working this problem can lead to several interesting class discussions.

I. Reasoning from assumptions

A) For students to do individually and in groups: Write the puzzle on the board. Have students try to (a) solve the puzzle and (b) convince other students that they are right.

B) As a whole class: Go over the puzzle together and have students share their thinking. An important step here is having students organize their thoughts and frame their arguments precisely as possible.

Students sometimes don't know how to begin. We can facilitate their thinking by asking, "Suppose we know that A is lying. What else do we then know?" Some students will respond that they know that B tells the truth (from statement 1). That means that C is indeed lying (from statement 2). But then either A or B must be telling the truth. We've assumed it's not A, so it must be B. So if A lies, B tells the truth and C lies, and we're OK.

It is important to note that this is not the only way to solve the problem. Students starting with some other assumption will eventually reach the same conclusion. We don't have to explicitly mention this to the students. But the idea that another argument will lead to the same conclusion is a good starter for conversation.

This is one possible solution to the puzzle, and students are typically satisfied with it. But it begs the question of whether it is the only possible solution. Perhaps B might be telling the truth, and one of the others might be lying. Let students enjoy their solution before raising this embarrassing question, which will be settled later in a more formal approach to the problem.

C) Many alternative solutions use the principle of *proof by contradiction*. If we make an assumption that leads to a contradiction (a statement or combination of statements that cannot be true), then the initial assumption must be wrong. Sometimes this sort of proof is called *indirect reasoning.*

Proof by contradiction can be difficult for students to understand. If some of the class doesn't understand, just proceed with those who do, and cycle back later to the notion of proof by contradiction. It will come up naturally in the solution of many of the puzzles here.

As an example of how proof by contradiction comes up naturally in this puzzle, suppose we start with the assumption that A is telling the truth. Then we know that B is lying. Now B asserts that C lies, so C must be telling the truth, which would mean that A lies. We have reached a contradiction: Starting with the assumption that A is telling the truth, we conclude that A lies. Thus our assumption is incorrect, and A must be lying. The rest of the solution proceeds as above.

Here's a slightly more complicated example of indirect reasoning that might emerge:

We assume that C is telling the truth. Then A and B both lie. In particular A lies. But by statement (1), this means that B is telling the truth, and C is lying. We started by assuming that C tells the truth, and end up deducing that C lies. This is a contradiction, so C cannot be telling the truth, i.e. C lies. Then B says C lies, so B must be telling the truth, and A is lying.

Similar reasoning holds if we assume that B lies, or B tells the truth, or if we assume that C lies, or C tells the truth.

There is no need to go over all of this explicitly in class. We mention these possibilities because the different arguments may come up in students' reasoning.

II. Structuring the reasoning

One interesting property of this sort of logic is that it is highly susceptible to automation. That is, when we structure the thinking, we can often find ways to make it a mechanical process. This eventually leads to the phenomenon of 'thinking machines': computers.

It is important not to proceed too quickly to this next stage in discussing the problem. It is useful for students to have a fluid and thoughtful conversation before the problem is made mechanical. The intuitions need to be developed before they are captured in formality.

Eventually, some students will find a way of writing down their reasoning. If students don't come up with one, show them the following way of structuring and recording their analysis. They can record the puzzle and set up the table in their journals. (DO NOT show them how to fill it in – that part is next.)

	A	B	C	Possible?
1.	T	T	T	No
2.	T	T	L	No
3.	T	L	T	No

4.	T	L	L	No
5.	L	T	T	No
6.	L	T	L	YES!
7.	L	L	T	No
8.	L	L	L	No

A) Individually – students construct the blank the table.

It is a worthwhile activity in itself for students to list all 8 possibilities. (Because there are 3 people and there are 2 possibilities for each, there are 2 x 2 x 2 total possibilities.)

It is useful to have a 'canonical' way of listing the possibilities. That is, once students understand that there are eight possibilities for truth tellers and liars, suggest to them that they be listed in the order given above (or any other order you choose, so long as it is the same for all students).

B) Use proof by contradiction to rule out possibilities: Have students consider each possibility and see whether it results in a contradiction. As they reason, they can fit their thinking into the chart to verify that there is only one possible answer.

Listing all the possibilities like this makes things much clearer, and shortens many arguments. For example:

Assume that B is lying.

B lies --> C tells the truth --> A lies --> B tells the truth (Contradiction!)

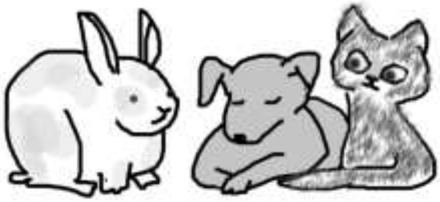
Now we can cross off *any* possibility on the table in which B is lying, and so on.

C) Class discussion: Make students aware of their use of “if...then” reasoning.

Make students aware that they are using “proof by contradiction.”

The will find themselves arguing, for example, “Suppose A is telling the truth – then what.
That can’t be because it leads to a contradiction . . .”

Next:

If exactly one of the following is true, which is it? **(a) Puppies are cuter than kittens.** **(b) Puppies are cuter than bunnies and kittens.** **(c) Puppies are by far the cutest animals.**	

If students don’t see it right away, give them the hint that the key phrase is “exactly one.”

From there we use a proof by contradiction.

If we assume (c) to be true, then (b) and (a) are true too. That contradicts the statement that only one is true.

If we assume (b) to be true, then (a) is true too. That also contradicts the statement that only one is true.

Therefore (a) is the only one that can be true because it is the only one that doesn’t lead to a contradiction.

This conclusion is enough for the problem to be worthwhile. But if a class is interested, we might go further. If (a) is the only statement that is true, then (c) must be false: there must be some animals cuter than puppies. And (b) must also be false, which means that bunnies are cuter than puppies. (Not every student will understand this deduction.) So we can infer a situation about animals, not just about statements: Bunnies are cuter than puppies, which are cuter than kittens.

Of course all this follows from the premise that exactly one of these statements is true—a premise which may have nothing to do with reality. Mathematical truth is logical, and not empirical.

If exactly one of the following is true, which is it?

(a) 10 year olds are delightful.
(b) 11 year olds are delightful.
(c) Either 10 year olds or 11 year olds are delightful.
(d) Blue is the loveliest color.

Solution: Again we use proof by contradiction.

If (a) is true, then (c) is also true, contradicting the statement that exactly one of the statements is true.
The same argument holds if (b) is true.
If (c) is true, then we know that at at least one of (a) or (b) is true.
Although we don't know which, we get a contradiction anyway:
more than one of the statements would have to be true.

So only (d) can be true.

BUILDING THE MEANING OF IMPLICATION

The fundamental task of any young student of logic is to get a handle on the notion of *implication*: of the way statement A can *imply* statement B. This is not so simple, and we are just looking at the easiest examples here.

We can use the language of implication by formulating the following principle with the students:

If statement (A) implies statement (B), then every time (A) is true, (B) must also be true. So in a list of statements, of which only one can be true, any statement in the list which *implies* another statement in the list must be false.

"APPLICATION" TO STANDARDIZED TESTS

This principle of logic actually can be 'applied' to the students' reality of the multiple choice test. In a well-constructed test of that form, there is exactly one right answer. That is, the list of answers is a list of statements, exactly one of which is true.

So, by the principle above, any choice which implies another choice must be wrong.

Bernard is thinking of a number.
(It is a whole number.)
Which one of the following is true:

(a) The number is greater than 7
(b) The number is greater than 8
(c) The number is greater than 12
(d) The number is greater than 15.

The solution hinges on the logical principle we outlined above. Statement (d) implies statement (c): if a number is greater than 15 it is certainly greater than 12. Similarly, (c) implies (b), and (b) implies (a). Thus (a) must be the correct answer.

Notice that we can tell something else. We know Bernard's number! Once we conclude that the number is a whole number greater than 7 but not greater than 8, we know it must in fact be 8.

Have students discuss this unusual phenomenon. It occurs not because we can read minds, but because Bernard is a character in a multiple choice test.

Class Discussion:

After students have worked through these puzzles, bring them together for a class discussion. What does this show about logic?

The main ideas here are:

Proof by contradiction. Suppose a particular statement implies a statement which must be false (because we know that some third statement is true). Then the statement we started with must in fact be false.

The essence of logical deduction is an "if-then" statement. (If A is true, then B must also be true.) We cannot expect a full formal understanding of this concept right now, but we can build towards it, by using the word 'implication' correctly to describe the relationship between two statements.

Logic is not about truth or falsehood of specific statements, but about relationships among statements. For this reason, logic can help us to derive new information from previous statements, and sometimes the results are surprising.

More Puzzles from Lewis Carroll

The puzzles below were included by Lewis Carroll in his book Symbolic Logic (1897). Carroll wrote the puzzles for adult students of formal logic, and he created his own system of symbols and diagrams for analyzing the puzzles. A few of the simpler examples are included here, which are accessible to young people and do not require the use of formal symbols. Students may find it useful to use Venn diagrams to analyze the problems, or they may invent their own symbols and diagrams. Notice that Carroll's puzzles include statements that silly, empirically unverifiable, or even false. The puzzles are amusing, of course, but Carroll also wanted to be sure that his students would solve the puzzles with pure logical reasoning and not with reference to any background knowledge.

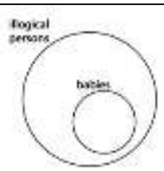
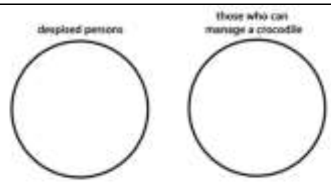
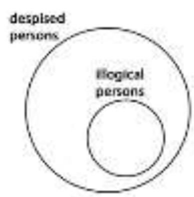
Worked example:

(a) All babies are illogical.
(b) Nobody is despised who can manage a crocodile.
(c) Illogical persons are despised.

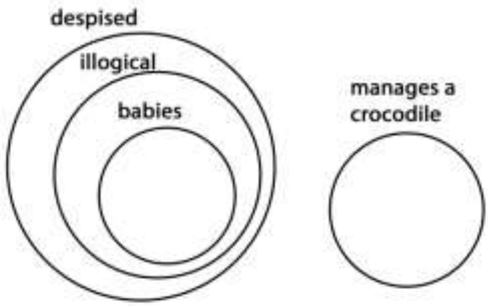
What can we conclude?

____

<u>Solution</u>: Each of the statements above can be represented with a Venn Diagram:

		
(a) All babies are illogical.	**(b) Nobody is despised who can manage a crocodile.**	**(c) Illogical persons are despised.**

And we can combine them:



Notice that, among other things, we can conclude:

There is no baby that manages a crocodile.
No illogical persons manage a crocodile.
All babies are despised.

Your Turn!

1. (a) No ducks waltz.
   (b) No officers ever decline to waltz.
   (c) All my poultry are ducks.

2. (a) All puddings are nice.
   (b) This dish is a pudding.
   (c) No nice things are wholesome.

3. (a) My saucepans are the only things I have that are made of tin.
   (b) I find all *your* presents very useful.
   (c) None of my saucepans are of the slightest use.

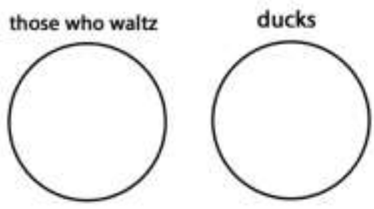
4. (a) No potatoes of mine, that are new, have been boiled.
(b) All my potatoes in this dish are fit to eat.
(c) No unboiled potatoes of mine are fit to eat.

5. (a) No one takes in the Times, unless he is well educated.
(b) No hedgehogs can read.
(c) Those who cannot read are not well educated.

6. (a) No birds, except ostriches, are 9 feet high.
(b) There are no birds in this aviary that belong to anyone but *me*.
(c) No ostrich lives on mince pies.
(d) I have no birds less than 9 feet high.

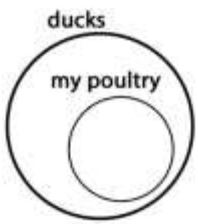
Solutions:

1. (a) No ducks waltz.

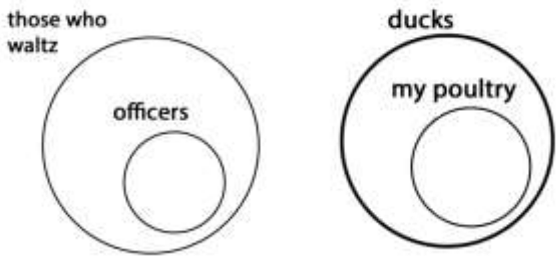


(b) No officers ever decline to waltz.

(c) All my poultry are ducks.



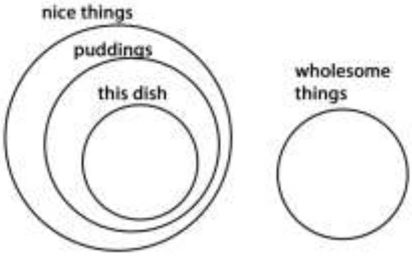
All together:



We can conclude, among other things, that my poultry are not officers and that they don't waltz. Also that no officer is a duck.

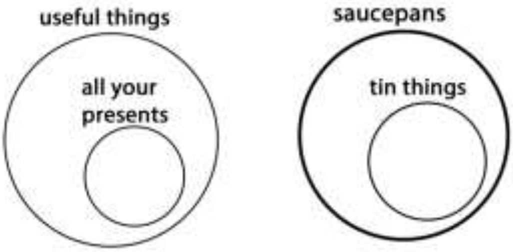
2. (a) All puddings are nice.
(b) This dish is a pudding.

(c) No nice things are wholesome.



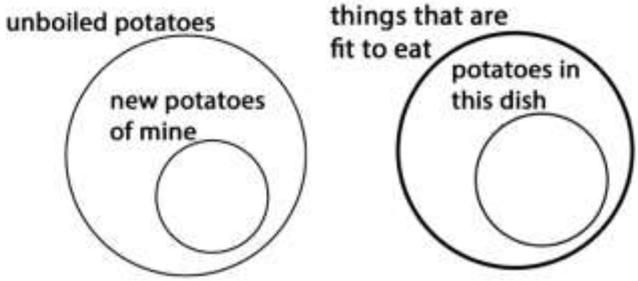
We can conclude, for example, that this dish is not wholesome.

3. (a) My saucepans are the only things I have that are made of tin.
(b) I find all *your* presents very useful.
(c) None of my saucepans are of the slightest use.



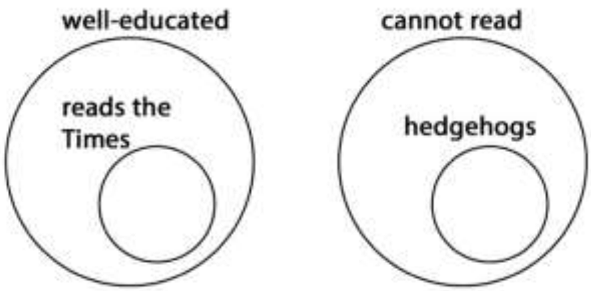
We can conclude, for example, that you never gave me anything made of tin.

4. (a) No potatoes of mine, that are new, have been boiled.
(b) All my potatoes in this dish are fit to eat.
(c) No unboiled potatoes of mine are fit to eat.



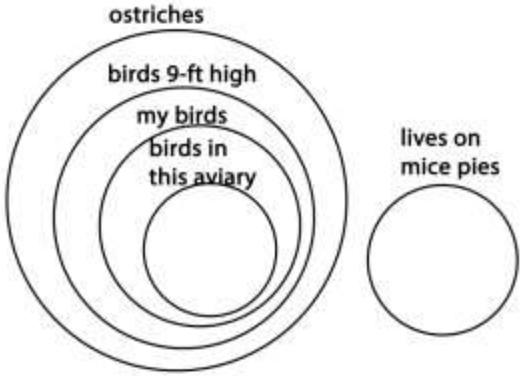
We can conclude, for example, that the potatoes in this dish have been boiled.

5. (a) No one takes in the Times, unless he is well educated.
(b) No hedgehogs can read.
(c) Those who cannot read are not well educated.



We can conclude, for example, that hedgehogs are not well educated.

6. (a) No birds, except ostriches, are 9 feet high.
(b) There are no birds in this aviary that belong to anyone but *me*.
(c) No ostrich lives on mince pies.
(d) I have no birds less than 9 feet high.



We can conclude, for example, that no birds in this aviary live on mince pies.

Cryptogram Puzzle

In this puzzle, each letter stands for another letter. Look for patterns in the words to figure out what the letters represent. For example, vowels occur more often than consonants. The location of commas and apostrophes can also give us clues. Students may also notice that the letters "iy" appear together a lot, and they can think about letter pairs that often go together. (In this case the letter pair represents "th.") The greatest tool we have here aside from looking for these sorts of patterns, is "proof by contradiction." If something seems reasonable, then assume it to be true and see if it leads to an impossibility.

One easy place to begin the solution is with the single letter B in the first panel. The letter stands by itself, and there are only two one-letter words in English: "a" and "I". If we assume that B stands for "a", we can easily reach a contradiction. For example, in the second panel, we have the code word B'N. But there is no contraction in English starting with the letter "a". So B cannot be "a" and must stand for I.

There are numerous paths to solution, all involving making use of logical deduction—together with the students' knowledge of English. (We have found that even students who are not native speakers of English can work this puzzle.)

*** Note: Normally we would see the letter "u" by itself and assume it must be an A or an I. Here, it is "*x*", which could lead students astray. It may be a good idea for the instructor to talk this out with the class and help them see that their assumption may not be true in this case. The letter “u” appears only one other place in the puzzle – at the end of a word. It must be a more unusual letter.

Solution:

Puzzle letter	Actual letter
B	I
C	R
D	A
E	J
F	S
G	B
H	K
I	T
L	U
M	D
N	M
P	E
Q	N
R	W
S	F
T	O
U	X
X	Y
Y	H

Solution:
Mrs. Smith, I know the answer! x is three!
No, I’m afraid the answer is fifty-six.
But Mrs. Smith, just yesterday you said x was three.